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ABOUT STOCHASTIC INTEGRALS WITH RESPECT TO PROCESSES WHICH ARE NOT SEMI-MARTINGALES

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1. Introduction

Let $(\Omega, \mathcal{F}_t, \mathbf{P})$ be a probability space with an increasing right continuous family of $(\mathcal{F}_\infty, \mathbf{P})$ -complete σ -algebras (\mathcal{F}_t) , and let \mathcal{P} be the predictable σ -algebra induced on $\Omega \times \mathbf{R}_+$ by the family (\mathcal{F}_t) .

For $H \in \mathcal{P}$, we write H_s for the random variable $\omega \rightarrow 1_H(s, \omega)$. If $Z = N + B$ is a semi-martingale such that N is a square integrable martingale and B an adapted process with square integrable variation, the mapping

$$(1) \quad H \rightarrow \int_0^\infty H_s dZ_s$$

defines a σ -additive vector measure on $(\Omega \times \mathbf{R}_+, \mathcal{P})$ with values in $L^2(\Omega, \mathcal{F}_\infty, \mathbf{P})$. It has been shown by several authors that conversely if μ is a σ -additive measure from \mathcal{P} to $L^2(\Omega, \mathcal{F}_\infty, \mathbf{P})$ given on the elementary predictable sets H of the form

$$H = h \times]s, t] \quad 0 < s < t, \quad h \in \mathcal{F}_s$$

by

$$(2) \quad \mu(H) = 1_h(Z_t - Z_s)$$

for a mean square right-continuous adapted process Z , then there is a modification of Z which is a semi-martingale [2].

Nevertheless, if we consider an other probability space $(W, \mathcal{W}, \mathbf{Q})$, an adapted process $(\omega, t) \rightarrow Z_t(\omega, w)$ depending on $w \in W$, and a measure μ which satisfies (2) for elementary predictable sets, and if we replace σ -additivity in $L^2(\mathbf{P})$ for each $w \in W$ by σ -additivity in $L^2(\mathbf{P} \times \mathbf{Q})$, it becomes possible that Z_t fails to be a semi-martingale for fixed w .

In the example that we give, Z_t is, for fixed w , the sum of a martingale and a process of zero energy similar to those considered by Fukushima [3] in order to give a probabilistic interpretation of functions in a Dirichlet space.

2. Random mixing of semi-martingales

Let $(U_\alpha(w))_{\alpha \in \mathbf{R}}$ be a second order process on $(W, \mathcal{W}, \mathcal{Q})$ which is right continuous in L^2 , with orthogonal increments and $\mathcal{B}(\mathbf{R}) \times \mathcal{W}$ measurable and let m be the positive Radon measure on \mathbf{R} associated to U_α by

$$m([\alpha, \beta]) = \mathbf{E}_Q(U_\beta - U_\alpha)^2, \quad \alpha < \beta.$$

Let $(M_t^\alpha(\omega))_{\alpha \in \mathbf{R}}$ be a family of right continuous and left limited martingales, and $(A_t^\alpha(\omega))_{\alpha \in \mathbf{R}}$ a family of continuous increasing adapted processes on $(\Omega, \mathcal{F}_t, \mathbf{P})$ such that the maps $(\alpha, \omega, s) \rightarrow M_s^\alpha(\omega)$ and $(\alpha, \omega, s) \rightarrow A_s^\alpha(\omega)$ are $\mathcal{B}(\mathbf{R}) \times \mathcal{F}_t \times \mathcal{B}(\mathbf{R}_+)$ measurable on $\mathbf{R} \times \Omega \times [0, t]$ and such that

$$(3) \quad \int_{\alpha \in \mathbf{R}} \mathbf{E}_P[(M_\infty^\alpha)^2 + (A_\infty^\alpha)^2] dm(\alpha) < +\infty.$$

Then we set $Z_t^\alpha(\omega) = M_t^\alpha(\omega) + A_t^\alpha(\omega)$ and

$$(4) \quad Z_t(\omega, w) = \int_{\alpha \in \mathbf{R}} Z_t^\alpha(\omega) dU_\alpha(w)$$

where the stochastic integral is of Wiener's type and exists for \mathbf{P} almost all ω since by (3) $Z_t^\alpha(\omega)$ belongs to $L^2(\mathbf{R}, \mathcal{B}(\mathbf{R}), dm(x))$ for \mathbf{P} -almost all ω .

For \mathbf{P} -almost ω the process $Z_t(\omega, w)$ is right continuous and left limited in $L^2(W, \mathcal{W}, \mathcal{Q})$.

If G is an elementary predictable process on $(\Omega, \mathcal{F}_t, \mathbf{P})$ given by:

$$G_s(\omega) = G_0(\omega) 1_{[0, t_1]}(s) + \dots + G_n(\omega) 1_{[t_n, t_{n+1}]}(s)$$

for $0 < t_1 < \dots < t_{n+1}$, where G_i is a \mathcal{F}_{t_i} -measurable bounded random variable, it follows immediately

$$G_0(Z_{t_1} - Z_0) + \dots + G_n(Z_{t_{n+1}} - Z_{t_n}) = \int_{\alpha \in \mathbf{R}} \left(\int_0^\infty G_s dZ_s^\alpha \right) dU_\alpha.$$

And we have:

Proposition 1. The map $H \in \mathcal{P} \rightarrow \int_0^t H_s dZ_s$ defined by

$$\int_0^t H_s dZ_s = \int_{\alpha \in \mathbf{R}} \left(\int_0^t H_s dZ_s^\alpha \right) dU_\alpha$$

is a σ -additive $L^2(\mathbf{P} \times \mathcal{Q})$ valued measure on $(\Omega \times \mathbf{R}_+, \mathcal{P})$.

Proof. Let $H^{(n)}$ be a sequence of disjoint predictable subsets of $\Omega \times \mathbf{R}_+$, we have

$$\begin{aligned} \mathbf{E}_P \mathbf{E}_Q \left[\int_{\alpha \in \mathbf{R}} \left(\int_0^t \sum_{n=N}^\infty H_s^{(n)} dZ_s^\alpha \right) dU_\alpha \right]^2 \\ = \int_{\alpha \in \mathbf{R}} \mathbf{E}_P \left(\int_0^t \sum_{n=N}^\infty H_s^{(n)} dZ_s^\alpha \right)^2 dm(\alpha) \end{aligned}$$

which can be made arbitrarily small for N large enough because

$$E_P \left(\int_0^t \sum_{n=N}^{\infty} H_s^{(n)} dZ_s^{\alpha} \right)^2$$

tends to zero and remains bounded by

$$2 E_P [(M_{\infty}^{\alpha})^2 + (A_{\infty}^{\alpha})^2] < +\infty. \quad \square$$

Set

$$Z_t^{(1)} = \int_{\alpha \in R} M_t^{\alpha} dU_{\alpha} \quad \text{and} \quad Z_t^{(2)} = \int_{\alpha \in R} A_t^{\alpha} dU_{\alpha}.$$

Lemma 2. *There is a $P \times Q$ -modification $\tilde{Z}_t^{(1)}$ of $Z_t^{(1)}$ which is a $(\Omega, \mathcal{F}_t, P)$ right continuous and left limited martingale for Q -almost all w .*

Proof. Let $G \in \mathcal{F}_s$, the following equalities hold in $L^2(W, \mathcal{W}, Q)$ for $s < t$:

$$\begin{aligned} E_P[1_G Z_t^{(1)}] &= \int_{\alpha \in R} E_P[1_G M_t^{\alpha}] dU_{\alpha} = \int_{\alpha \in R} E_P[1_G M_s^{\alpha}] dU_{\alpha} \\ &= E_P[1_G Z_s^{(1)}], \end{aligned}$$

therefore, if we choose a $\mathcal{F}_t \times \mathcal{W}$ -measurable element $z_t^{(1)}(\omega, w)$ in the $L^2(P \times Q)$ equivalence class of $Z_t^{(1)}$, for w outside a Q -negligible set \mathcal{N} , $z_s^{(1)}$ is a (\mathcal{F}_s, P) -martingale for rational s .

Then, if we put $\tilde{Z}_t^{(1)} = \lim_{\substack{s \text{ rational} \\ s \downarrow t}} z_s^{(1)}$, for $w \notin \mathcal{N}$, $\tilde{Z}_t^{(1)}$ is P -almost surely a right

continuous and left limited (\mathcal{F}_t) -martingale and

$$\tilde{Z}_t^{(1)} = Z_t^{(1)} \quad P \times Q\text{-a.e.}$$

because $Z_t^{(1)}$ is right continuous in $L^2(P \times Q)$. \square

As concerns $Z_t^{(2)}$, it is a zero energy process:

Lemma 3. *Let τ_n be a sequence of partitions of $[0, t]$ with diameter tending to zero, then*

$$E_Q E_P \left[\sum_{t_i \in \tau_n} (Z_{t_{i+1}}^{(2)} - Z_{t_i}^{(2)})^2 \right] \xrightarrow{n \rightarrow \infty} 0.$$

Proof. The expression is equal to

$$E_P \int_{\alpha \in R} \sum_{\tau_n} (A_{t_{i+1}}^{\alpha} - A_{t_i}^{\alpha})^2 dm(\alpha),$$

and $\sum_{\tau_n} (A_{t_{i+1}}^{\alpha} - A_{t_i}^{\alpha})^2$ tends to zero, because A_t^{α} is continuous, and remains majorized by $(A_{\infty}^{\alpha})^2$, which gives the result by (3).

Nevertheless, in general $Z_t^{(2)}$ has no modification with finite variation, as shown by the following example:

Let X be a continuous martingale on $(\Omega, \mathcal{F}_t, \mathbf{P})$ such that

$$E_{\mathbf{P}} X_{\infty}^2 < +\infty.$$

Let

$$M_t^{\alpha} = \int_0^t 1_{\{X_s > \alpha\}} dX_s$$

and $A_t^{\alpha} = \frac{1}{2} L_t^{\alpha}$

where L_t^{α} is the local time of X at α . Condition (3) is satisfied as soon as the measure m is finite. If we put

$$Z_t = \int_{\alpha \in \mathbf{R}} M_t^{\alpha} dU_{\alpha} + \int_{\alpha \in \mathbf{R}} A_t^{\alpha} dU_{\alpha}$$

we have, from Meyer-Tanaka's formula:

$$Z_t = \int_{\alpha \in \mathbf{R}} [(X_t - \alpha)^+ - (X_0 - \alpha)^+] dU_{\alpha} = \int_{x_0}^{x_t} U_{\lambda} d\lambda \quad \mathbf{P} \times \mathbf{Q} \text{ a.e.}$$

If Z_t had a $\mathbf{P} \times \mathbf{Q}$ -modification such that, for fixed $w \in W$, \tilde{Z}_t were a $(\Omega, \mathcal{F}_t, \mathbf{P})$ semi-martingale, then, since \tilde{Z}_t and $\int_{x_0}^{x_t} U_{\lambda} d\lambda$ are both right continuous, $\int_{x_0}^{x_t} U_{\lambda} d\lambda$ would be a semi-martingale. So, from ([1], theorem 5, 6), if we took for X a real stopped brownian motion starting at 0, the map

$$x \xrightarrow{\psi} \int_0^x U_{\lambda}(w) d\lambda$$

would be the difference of two convex functions. But, if for example, U itself is a stopped brownian motion, that can be true only on a \mathbf{Q} -negligible set because almost all brownian sample paths have not finite variation. So, in this case, the $\mathbf{P} \times \mathbf{Q}$ -modifications of Z_t are \mathbf{Q} a.e. not semi-martingales.

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